Implicit Solution of the Vertical Diffusion Equation for the Reynolds Stress Mixing Models

The solution procedure in HYCOM follows the procedure used by Large *et al.* (1994) and is used for all three Reynolds stress vertical mixing models (KPP, GISS, MY). Decomposing model variables into mean (denoted by an overbar) and turbulent (denoted by a prime) components, the vertical diffusion equations to be solved for potential temperature, salinity, and vector momentum are

$$\frac{\partial \overline{\boldsymbol{q}}}{\partial t} = -\frac{\partial}{\partial z}\overline{w'\boldsymbol{q}'} \quad \frac{\partial \overline{S}}{\partial t} = -\frac{\partial}{\partial z}\overline{w'S'} \quad \frac{\partial \overline{\mathbf{v}}}{\partial t} = -\frac{\partial}{\partial z}\overline{w'\mathbf{v}'}. \tag{1}$$

Boundary layer diffusivities and viscosity parameterized as follows:

$$\overline{w'\boldsymbol{q}'} = -K_q\left(\frac{\partial \overline{\boldsymbol{q}}}{\partial z} + g_q\right), \quad \overline{w'S'} = -K_S\left(\frac{\partial \overline{S}}{\partial z} + g_S\right), \quad \overline{w'\mathbf{v}'} = -K_m\left(\frac{\partial \overline{\mathbf{v}}}{\partial z} + g_m\right), \tag{2}$$

where the g terms represent nonlocal fluxes. For example, the KPP model includes nonlocal terms for $\boldsymbol{q}$ and S, but not for momentum. The following solution procedure is valid for any mixing model in HYCOM that calculates the diffusivity/viscosity profiles at model interfaces, whether or not nonlocal terms are parameterized.

The following matrix problems are formulated and solved:

$$\mathbf{A_T}\Theta^{t+1} = \Theta^t + \mathbf{H}_\Theta \quad \mathbf{A_S}\mathbf{S}^{t+1} = \mathbf{S}^t + \mathbf{H_S} \quad \mathbf{A_M}\mathbf{M}^{t+1} = \mathbf{M}^t + \mathbf{H_M}, \tag{3}$$

where superscripts $t, t+1$ denote model times, and $\mathbf{M}$ is the vector of a momentum component, either u or v. The matrices $\mathbf{A}$ are tri-diagonal coefficient matrices, while the vectors $\mathbf{H_T}$ and $\mathbf{H_S}$ represent the nonlocal flux terms. Given K model layers with nonzero thickness, where an individual layer k of thickness $\boldsymbol{d}p_k$ is bounded above and below by interfaces located at pressures p_k and p_{k+1}, the matrix $\mathbf{A_S}$ is determined as follows:

$$\begin{aligned}
\mathbf{A_S^{1,1}} &= \left(1 + \Omega_{S1}^{+}\right) & & \\
\mathbf{A_S}^{k,k-1} &= -\Omega_{Sk}^{-} & & 2 \le k \le \mathrm{K} \\
\mathbf{A_S}^{k,k} &= \left(1 + \Omega_{Sk}^{-} + \Omega_{Sk}^{+}\right) & & 2 \le k \le \mathrm{K} \quad , \\
\mathbf{A_s}^{k,k+1} &= -\Omega_{Sk}^{+} & & 1 \le k \le \mathrm{K}-1
\end{aligned} \tag{4}$$

with

$$\begin{aligned}
\Omega_{Sk}^{-} &= \frac{\Delta t}{\boldsymbol{d}p_k}\frac{K_S\left(p_k\right)}{\left(p_{k+0.5} - p_{k-0.5}\right)} \\
\Omega_{Sk}^{+} &= \frac{\Delta t}{\boldsymbol{d}p_k}\frac{K_S\left(p_{k+1}\right)}{\left(p_{k+1.5} - p_{k+0.5}\right)}
\end{aligned}, \tag{5}$$

where $p_{k+0.5}$ represents the central pressure depth of model layer k. The nonlocal flux arrays are calculated using

$$
\begin{aligned}
H_{S1} &= \frac{\Delta t}{\delta p_1} K_S\left(p_{k+1}\right)\gamma_S\left(p_{k+1}\right) \\
H_{Sk} &= \frac{\Delta t}{\delta p_k}\left[K_S\left(p_{k+1}\right)\gamma_S\left(p_{k+1}\right) - K_S\left(p_k\right)\gamma_S\left(p_k\right)\right] \quad 2 \le k \le \mathrm{K}\,.
\end{aligned}
\tag{6}
$$

The solution is then found by inverting the tri-diagonal matrix $\mathbf{A}$. The matrix problems are formulated and solved in the same manner for potential temperature and momentum components.

REFERENCE

Large, W. G., J. C. Mc Williams, and S. C. Doney, 1994: Oceanic vertical mixing: a review and a model with a nonlocal boundary layer parameterization. *Rev. Geophys*. **32**, 363-403.