The Momentum Balance in HYCOM

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General Remarks

Except for the vertical diffusion term, the momentum equation in HYCOM is the same as in MICOM with a few exceptions. The momentum balance has, of course, been modified to handle horizontally varying density in all model layers. Surface momentum flux accelerates only the fluid in layer 1 except when the Kraus-Turner mixed layer model Is used, when the momentum change is distributed over the full mixed layer depth that was calculated during the previous time step.

The most significant modification of the momentum equation in hybrid vertical coordinates concerns the horizontal pressure gradient force (PGF), which is discussed in detail below.

The Horizontal Pressure Gradient Force in Generalized Coordinates

In a hydrostatic fluid ($\partial j / \partial s = -a \partial p / \partial s$), where f is the geopotential, a is the specific volume, and s is the generalized vertical coordinate, the layer mass-weighted PGF satisfies

$$\frac{\partial p}{\partial s}\left[a \nabla_s p + \nabla_s f\right] = \nabla_s\left(\frac{\partial p}{\partial s} a\, p\right) + \frac{\partial}{\partial s}\left(p \nabla_s f\right). \qquad (1)$$

The three-dimensional gradient form on the right indicates that only boundary forces can cause net accelerations of the fluid system. This has well known implications for vortex spinup and spindown. Given that the curl of the right side of (1) reduces to

$$\frac{\partial}{\partial s}\left(\nabla_s \times \nabla_s f\right),$$

the interface pressure torques governing vortex spinup and spindown in individual s coordinate layers have the form $\left(\nabla_s \times p \nabla_s f\right)$. It is important to preserve these properties when numerically solving the fluid dynamics equations.

It is therefore necessary to find a finite-difference expression for the PGF term $\left[a \nabla_s p + \nabla_s f\right]$ in the horizontal momentum equation that can, after multiplication by layer pressure thickness, be transformed by finite difference operations into an analog of the right side of (1). To illustrate this, the x component of the rightmost term of (1) is written in the simplest possible, and thus plausible, form $d_s\left(\overline{p}^x d_x f\right)$. Finite difference product differentiation rules allow this to be expanded as follows:

$$d_s\left(\overline{p}^x d_x f\right) = \left(d_s \overline{p}^x\right) d_x \overline{f}^x + \overline{p}^{xs} d_x d_s f$$

$$d_s\left(\overline{p}^x d_x f\right) = \left(d_s \overline{p}^x\right) d_x \overline{f}^x - \overline{p}^{xs} d_x\left(a d_s p\right)$$

$$d_s\left(\overline{p}^x d_x f\right) = \left(d_s \overline{p}^x\right) d_x \overline{f}^x - d_x\left(\overline{p}^s a d_s p\right) + \overline{a d_s p}^x d_x \overline{p}^s$$

A finite difference equation analogous to (1) is obtained by rearranging terms and adding the analogous equation for the y component:

$$\begin{aligned} \overline{a d_s p}^x d_x \overline{p}^s + \left(d_s \overline{p}^x\right) d_x \overline{f}^s &= d_x\left(\overline{p}^s a d_s p\right) + d_s\left(\overline{p}^x d_x f\right) \\ \overline{a d_s p}^y d_y \overline{p}^s + \left(d_s \overline{p}^y\right) d_y \overline{f}^s &= d_y\left(\overline{p}^s a d_s p\right) + d_s\left(\overline{p}^y d_y f\right) \end{aligned}.$$

To preserve the conservation properties expressed by (1), the finite-difference PGF term must be evaluated in the form

$$a\nabla_s p + \nabla_s f = \begin{pmatrix} \dfrac{\overline{a d_s p}^x}{\overline{d_s p}^x} d_x \overline{p}^s + d_x \overline{f}^s \\ \dfrac{\overline{a d_s p}^y}{\overline{d_s p}^y} d_y \overline{p}^s + d_y \overline{f}^s \end{pmatrix}. \qquad (2)$$

The salient result of this analysis is that writing the undifferentiated factor a in the PGF formula as simply $\overline{a}^x$ or $\overline{a}^y$ can lead to spurious momentum and vorticity generation. To avoid this pitfall, a must be weighted by layer thickness in the PGF formula.

For use in isopycnic or quasi-isopycnic models, it is convenient to express the PGF in terms of the Montgomery potential $M = f + pa$. The proper finite difference analog of M in a staggered vertical grid (p and f carried on layer interfaces and a carried within layers) is

$$M = \overline{f}^s + a\,\overline{p}^s.$$

The identity $d_s\left(a\,\overline{p}^s\right) = \overline{a d_s p}^s + p d_s a$ enables the s derivative of M to be expanded into

$$d_s M = p d_s a + \overline{d_s f + a d_s p}^s,$$

from which the finite difference analogs of the two common forms of the hydrostatic equation can be extracted:

$$\frac{\partial f}{\partial p} = -a \rightarrow d_s f = -a d_s p$$

$$\frac{\partial M}{\partial a} = p \rightarrow d_s M = p d_s a$$

The x component of (2) becomes

$$\frac{\overline{a d_s p}^x}{\overline{d_s p}^x} d_x \overline{p}^s + d_x \overline{f}^s = d_x M + \left[\frac{\overline{a d_s p}^x}{\overline{d_s p}^x} d_x \overline{p}^s - d_x\left(a\,\overline{p}^s\right)\right]. \qquad (3)$$

Making use of the relation

$$\overline{AB}^x - \overline{A}^x \overline{B}^x = \frac{1}{4}\left(d'_x A\right)\left(d'_x B\right),$$

where d'_x represents the difference between two neighboring grid points; i.e., $d'_x = \Delta x d_x$, the term within square brackets in (3) can be expanded into

$$\frac{1}{\overline{d_s p}^x}\left(\overline{a d_s p}^x - \overline{a}^x \overline{d_s p}^x\right) d_x \overline{p}^s - \overline{p}^{sx} d_x a = \frac{1}{4\overline{d_s p}^x}\left(d'_x d_s p\right)\left(d'_x a\right) d_x \overline{p}^s - \overline{p}^{sx} d_x a$$
$$\frac{1}{\overline{d_s p}^x}\left(\overline{a d_s p}^x - \overline{a}^x \overline{d_s p}^x\right) d_x \overline{p}^s - \overline{p}^{sx} d_x a = \frac{1}{4\overline{d_s p}^x}\left[\left(d'_x d_s p\right)\left(d'_x \overline{p}^s\right) - 4\overline{p}^{sx} d_s \overline{p}^x\right] d_x a.$$

The above expression involves a total of four p points located one grid distance Δx apart on two consecutive s surfaces. Substantial simplification is possible by labeling the four grid points as

$$p_1 = p\left(x - \frac{\Delta x}{2}, s - \frac{\Delta s}{2}\right) \quad p_2 = p\left(x + \frac{\Delta x}{2}, s - \frac{\Delta s}{2}\right)$$
$$p_3 = p\left(x - \frac{\Delta x}{2}, s + \frac{\Delta s}{2}\right) \quad p_4 = p\left(x + \frac{\Delta x}{2}, s + \frac{\Delta s}{2}\right).$$

With a modest amount of algebra, the term within square brackets in (3) reduces to

$$\frac{p_1 p_2 - p_3 p_4}{\left(p_4 - p_2\right) + \left(p_3 - p_1\right)} d_x a.$$

Substitution of this expression inside the brackets of (3) yields the sought-after expression for the x component of the PGF force. An analogous can be derived for the y component.